Geometric diagrams in LaTeX

Peter J. Cameron

1 Introduction

Teachers of geometry since Euclid have known the value of diagrams to help students in following geometric arguments. Paradoxically, it is necessary that the argument be capable of standing alone, without the diagram — Euclid himself realised the possibility of reaching false conclusions from reasoning depending on a faulty diagram — but an accurately-drawn diagram can be a valuable aid to following the reasoning.

LaTeX provides tools (in the `picture` environment) for drawing accurate and clear diagrams, which are a great improvement on what I can produce by hand. How adequate are these tools for complicated diagrams?

An important feature of diagrams for pedagogical purposes is that they should be *generic*. For example, no two lines should be parallel, and no triangle isosceles, unless this is part of the specification of the diagram or a geometric consequence of it. Such unwanted special features could create the misleading impression that they are necessary for the conclusion.

Of course, more powerful (and cumbersome) picture-drawing tools are available; but LaTeX has the advantage that it is always there. Also, as we will see, the investigation throws up some interesting problems!

2 Triangles

The `picture` environment enables line segments with any one of 48 different slopes to be drawn. The allowable slopes are 0 (horizontal), ∞ (vertical), any ratio x/y where x and y lie between 1 and 6 inclusive, and the negatives of these.

Three different slopes determine uniquely the shape of a triangle. We are still free to determine the size and position of the triangle by positioning the line segments appropriately.

It is easy to draw an isosceles triangle in which the base is horizontal or vertical (the other two slopes a, b should satisfy $b = -a$) or makes an angle of $45°$ with the axes (the other slopes satisfy $ab = 1$). Curiously, it turns out that any other isosceles triangle which can be drawn is right-angled (with base angles of $45°$). There appears to be no logical reason for this fact, which is established by checking all possibilities. A typical example has slopes of $1/5$ (for the base), $-2/3$ and $3/2$. Up to reflection in the axes, there are twelve such triangles.

Note in passing that it is not possible to draw an equilateral triangle accurately in LaTeX, for all allowable slopes are rational numbers, and so the tangents of the angles between them are also rational; but

$$\tan 60° = \sqrt{3}$$

is irrational. Tolerable approximations can be produced; for example, $5/3$ differs from $\sqrt{3}$ by only about 4%.

On the other hand, rectangles and squares can be drawn in many different orientations, since lines with slopes a and $-1/a$ are necessarily perpendicular. The interested reader may like to consider other metric properties of quadrilaterals and higher polygons.

3 Quadrangles

The last section was about *Euclidean geometry*, concerning actual values of lengths and angles. (In practice this means that properties will be destroyed if the horizontal and vertical resolution of the output device are not equal; for example, isosceles triangles will no longer be isosceles). In what follows, I will be concerned with *affine geometry*, which does not have this defect. Only incidence and parallelism, and properties derived from these are significant. For example, we can say that two line-segments have the same direction; if they have the same direction (but not otherwise), it is meaningful to say that they have the same length.

The first problem that arises is to draw a *complete quadrangle*, consisting of four points and all six lines joining them. Here, the slopes of the lines determine the figure up to choice of position and scale; but the six slopes are not independent. If they are a, b, c, d, e, f, where the pairs a and b, c and d, e and f are "opposite", and a, b and c pass through one of the points, then

$$(b-c)(d-e)(f-a) = (a-d)(b-e)(c-f).$$

Using this formula, f can be expressed in terms of the other five slopes. Of course, not all choices of $a, \ldots, e$ will give an allowable value for f. Determining the allowable possibilities is clearly work for computers!

A note on this computation. It is an advantage to do all the calculations in integers. If the non-zero LaTeX slopes are multiplied by 60, they become integers, and can be recognized by the fact that they are all the divisors of 3600 between 10 and 360 inclusive, except for 16, 18, 25, 144, 200, 225 and their negatives. Moreover, since our equation is homogeneous, it remains true after this multiplication. The

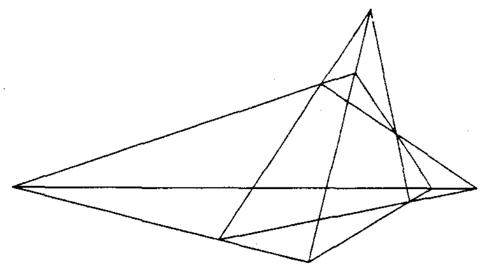

Figure 1: Desargues' Theorem.

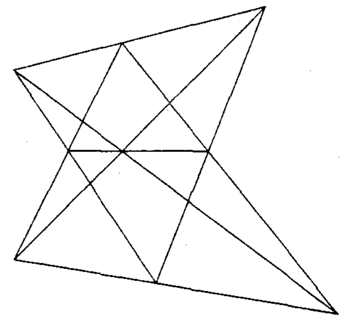

Figure 2: Pappus' theorem.

good news to emerge from the computation is that there are far too many solutions to list here!

4 Desargues and Pappus

Two very important theorems of affine geometry are those of Desargues and Pappus. Each has the property that it is cumbersome and unenlightening to state in words, but simply expressed by a diagram. Desargues' theorem states that if we have the ten points and nine of the ten lines of Figure 1, then the tenth line also occurs (that is, the three corresponding points are collinear). Pappus' theorem has a similar diagram with nine points and nine lines (see below).

If we look at the Desargues configuration, we see five complete quadrangles in it. So the work done in the last section can form the basis of a computer search for suitable parameters: seven slopes will determine the shape of the figure. Before I made this search, I spent some time looking for solutions by trial and error, without success. The surprising result of the exhaustive search was that there are many tens of thousands of solutions. The particular one used to draw Figure 1 is as follows. It is chosen to be reasonably "generic", in the earlier sense.

```
\setlength{\unitlength}{0.3mm}
\begin{picture}(220,130)(-10,-40)
\put(0,0){\line(1,0){207}}
\put(0,0){\line(3,1){153}}
\put(0,0){\line(4,-1){132}}
\put(92,-23){\line(2,3){68}}
\put(92,-23){\line(5,1){115}}
\put(132,-33){\line(5,3){55}}
\put(132,-33){\line(1,4){28}}
\put(177,-6){\line(-1,5){17}}
\put(187,0){\line(-2,3){34}}
\put(207,0){\line(-3,2){69}}
\end{picture}
```

Pappus' theorem is illustrated in Figure 2.

Unlike the last one, this figure contains no complete quadrangles, so we have to start its analysis from scratch.

There is an added difficulty. The equation connecting the slopes of a complete quadrangle has degree 3; so individual terms cannot exceed $360^3 = 46656000$; using 32-bit integers, there is no risk of overflow. However, in the case of Pappus, we have a fifth-degree equation (even after assuming that one line is horizontal), and integer overflow is inevitable. This can be alleviated to some extent by dividing by common factors wherever possible; but there were still many cases when the terms grew too large. So I abandoned the exhaustive search and simply looked for what I could find.

The general picture was the same as for Desargues: there are thousands of acceptable solutions, but it is very difficult to find even one by trial and error.

5 Conclusion

LaTeX provides tools for drawing quite complicated diagrams. But there are some things that can't be done, and others where you have to work quite hard (and even resort to computation) to find the right way to draw your figure. Readers are encouraged to try their luck with other geometric configurations. (I have concentrated on some of the figures I had to draw for a volume of lecture notes on geometry.)

Perhaps someone will be inspired to write a program which does all this automatically. The input should be the specification of the diagram, with optionally some comments about desirable or undesirable coincidences of length or slope. The program should either produce LaTeX input for an acceptable figure, or tell the operator that it can't be done.

◇ Peter J. Cameron
School of Mathematical Sciences
Queen Mary and Westfield College
Mile End Road
London E1 4NS
U.K.
Janet: pjc@uk.ac.qmw.maths